

# A Note about the Chain Rule

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I wrote this note in order to clarify to myself some usage of the chain rule I was not familiar with (or had forgotten over the years). I thought this might help others so I wrote it up.

## Using the 2-dimensional Chain Rule

CHAIN RULE

$$\frac{\partial}{\partial \cdot} = \frac{\partial x}{\partial \cdot} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \cdot} \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \cdot} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \cdot}$$

For example:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad \text{where } z = f(x, y) \quad \text{and} \quad x = f(t), y = f(t)$$

Suppose that we have  $f(U, \frac{\partial U}{\partial \tau}, \frac{\partial U}{\partial \xi}, \frac{\partial^2 U}{\partial \xi^2})$  where  $U = U(\xi, \tau)$  and that we want to change variables to obtain a function of  $(x, t)$ :

$$U(\xi, \tau) = W(x, t)$$

Establish the change of variables:

$$\begin{aligned} x &= f_1(\xi, \tau) & t &= f_2(\xi, \tau) & \text{then} \\ \xi &= g_1(x, t) & \tau &= g_2(x, t) \end{aligned}$$

$$dU = dx \frac{\partial U}{\partial x} + dt \frac{\partial U}{\partial t} \rightarrow \frac{\partial U}{\partial \cdot} = \frac{\partial x}{\partial \cdot} \frac{\partial U}{\partial x} + \frac{\partial t}{\partial \cdot} \frac{\partial U}{\partial t} \rightarrow \frac{\partial}{\partial \cdot} = \frac{\partial x}{\partial \cdot} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \cdot} \frac{\partial}{\partial t}$$

Apply this to  $\xi, \tau$  to get the operators for our change of variables:

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi} \frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial \tau} = \frac{\partial x}{\partial \tau} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \tau} \frac{\partial}{\partial t}$$

### Example (as seen in the Black-Scholes derivation)

$$\frac{\partial U}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 U}{\partial \xi^2} + K \frac{\partial U}{\partial \xi}$$

We want to convert  $(\xi, \tau)$  to  $(x, t)$ :

$$U(\xi, \tau) = W(x, t)$$

where we wish to apply a translation

$$x = \xi + K\tau \quad \text{and} \quad t = \tau$$

in order to get rid of the  $K$  term.

We find the partial derivatives for our parameters:

$$\frac{\partial x}{\partial \xi} = 1 \quad \frac{\partial t}{\partial \xi} = 0 \quad \frac{\partial x}{\partial \tau} = K \quad \frac{\partial t}{\partial \tau} = 1$$

Then specialize the chain rule for your parameters:

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi} \frac{\partial}{\partial t} = 1 \frac{\partial}{\partial x} + 0 \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \quad (1)$$

$$\frac{\partial}{\partial \tau} = \frac{\partial x}{\partial \tau} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \tau} \frac{\partial}{\partial t} = K \frac{\partial}{\partial x} + 1 \frac{\partial}{\partial t} = K \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \quad (2)$$

And apply the operators to our function  $U$  to get the equivalence in terms of  $W$ ,  $x$  and  $t$ :

$$\frac{\partial U}{\partial \tau} = K \frac{\partial W}{\partial x} + \frac{\partial W}{\partial t} \qquad \frac{\partial U}{\partial \xi} = \frac{\partial W}{\partial x} \qquad \frac{\partial^2 U}{\partial \xi^2} = \frac{\partial^2 W}{\partial x^2}$$

Replacing into our original equation, one term cancels out and we obtain a Heat equation (with a known solution):

$$\cancel{K} \frac{\partial W}{\partial x} + \frac{\partial W}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 W}{\partial x^2} + \cancel{K} \frac{\partial W}{\partial x} \quad (3)$$

$$\frac{\partial W}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 W}{\partial x^2} \quad (4)$$

## Chain Rule with Exponential/Logarithm

Another chain rule trickiness that seems to twist with my mind a bit funny is the usage of an exponential/logarithmic change of variable.

$$S = e^{-\xi} \qquad \frac{\partial}{\partial S} = \frac{\partial \xi}{\partial S} \frac{\partial}{\partial \xi} = \frac{1}{S} \frac{\partial}{\partial \xi} = e^{\xi} \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial \xi} = S \frac{\partial}{\partial S} = e^{-\xi} \frac{\partial}{\partial S}$$

## Chain Rule with Exponential/Logarithm (II)

This is a great trick to reduce expressions to a total derivative. In the following  $f : f(x)$ ,  $g : g(x)$ ,  $h : h(x)$ :

$$d(f \cdot g) = df \cdot g + f \cdot dg$$

$$g^{-1} d(f \cdot g) = df + f \cdot \frac{dg}{g}$$

By putting  $g(x) = e^{h(x)}$ ,

$$dg = g \cdot dh = e^{h(x)} dh$$

$$e^{-h(x)} d(e^{h(x)} \cdot f) = df + f \cdot dh$$

e.g.

$$dV_i - dV_a - (V_i - V_a)r dt$$

$$d(V_i - V_a) + (V_i - V_a)(-r dt)$$

$$e^{rt} d(e^{-rt}(V_i - V_a))$$